The decline in the British bank population since 1810 obeys a law of negative compound interest

Two models by Garnett *et al.* [*J. Bus. Hist.* **57**(1):182-202 (2015)] for the organisational demography of British banks are explored analytically: the exponential model; and the agent-based system (ABM) governed by probabilistic interactions. Exact expressions for ABM expectation values are derived, revealing first that bank creations obey a 'birth' process, and second, that one of the ABM hypotheses may be discarded. The expectation values are used to demonstrate that beneath its stochastic implementation, the ABM model is a discrete analogue of the exponential model, meaning that the decline in the British bank population obeys a law of negative compound interest.

Keywords: banking; agent-based modelling; simulation; recurrence relation

Subject classification codes: include these here if the journal requires them

1. Introduction

A recent article by Garnett *et al.* presents an intriguing study of the decline in the number of British banks (or 'bank population') since 1810, based on two quantitative models of the authors' comprehensive and newly compiled data series (Garnett 2015). The authors' approach is valuable and compelling, particularly in its creative use of multi-disciplinary techniques to construct a computational agent-based model of bank demography, and thence describe the decrease in size of the population with recourse to a minimal number of mechanisms. Indeed, one of the strengths of the article is that it prompts a number of connected research questions: for instance, is it possible to derive closed form expressions which support the authors' numerical study?; can the agent based system be further simplified?; and is there a way of linking the two quantitative models together despite ostensibly dissimilar modelling assumptions? Here I attempt to address these questions, and in so doing will derive a number of new analytical expressions for features of the agent-based system. Note that in some places I shall have an opportunity to refine the authors' approach, and in this respect my results should seen as complementing and contributing to their main conclusions.

The first model described by Garnett *et al.* takes the form of a differential equation governing the number of banks $B(t)$ at time t, *viz*

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = \lambda B(t), \quad \text{where} \quad \lambda = (X - Y), \tag{1.1}$$

with X and Y operating as 'birth' and 'death' rates respectively. This equation has the well known solution

$$B(t) = B(0)exp(\lambda t), \tag{1.2}$$

where $B(0)$ is the size of the population at time t=0. It will be clear from the form of this solution that the population undergoes exponential increase when births exceed deaths, i.e, X>Y (λ>0), and exponential decrease when deaths exceed births, that is, X<Y (λ<0). During the period of decline, a simple exponential fit to the historical data yields $\lambda = -1.0\times10^{-3}$ per month, with an R^2 value of 0.96 (see figure 1).

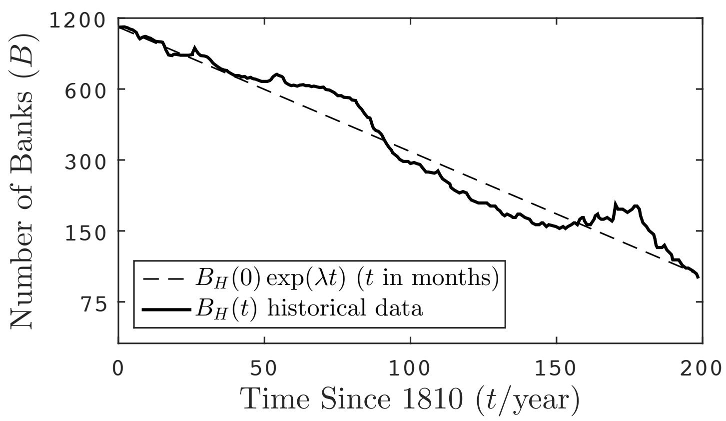

Figure 1. Comparison between the historical data series and the exponential model of equation (1.1) assuming $\lambda = -1.0\times10^{-3}$ per month. The historical data has been extracted from Garnett *et al.* (2015).

Arguing that the form of equation (1.2) represents more of a description of the decline in the British bank population than a mechanistic picture of the process, Garnett *et al.* propose an alternative model for the system based on an agent-based simulation. In this second model banks are created with some probability *a* per month, are permitted to fail with probability *b* per month, and (if they don't fail) undergo some kind of pairwise merger process with probability *c* per month. In addition, merger processes are assumed to have a finite duration, concluding with probability *d* per month, where a given merger must finish before the banks involved are permitted to begin amalgamation anew (see table below). Note that for the majority of their article Garnett *et al.* treat the banks themselves as being indistinguishable, that is, the probabilities associated with each kind of processes are the same, regardless of bank age or size. The authors' relax this assumption towards the end of their paper; however, I shall adopt it here throughout.

Table 1. Probabilities used in the agent-based model (Garnett *et al.* 2015).

Event	**Probability per bank per time step (month^{-1})**
Bank creation (*a*)	0.0013
Bank failure (*b*)	0.0012
Begin merger process (*c*)	0.0023
Conclude merger process (*d*)	0.4

Further details concerning the authors' model will be discussed in later sections, but before proceeding it is worth motivating other aspects of the current article by making some remarks about agent-based modelling more generally. In particular, notice that the type of model investigated by the authors is an example of a stochastic system: individual banks (agents) interact according to probabilities, so that the output from any given simulation depends to some extent on chance. Indeed, even if the initial conditions are identical, no two simulation results will be exactly alike. When attempting to explore features of a stochastic model, therefore, reliance on sample simulations can quickly become problematic. In particular, if we observe a discrepancy between the output from a sample simulation and the system we are modelling, then we are faced with the following predicament: does the discrepancy imply a fundamental difference between the model and the system, or has it arisen due to an artefact of the sample simulation in question?

Fortunately, for many stochastic models (including that investigated by Garnett *et al.*) individual data points from sample simulations cluster around a well defined path according to the system's *expectation values*: a set of variables defined in terms of the mean values calculated from n_s samples in the limit $n_s \to \infty$, but in practice approximated by the mean with n_s finite (this approximation is usually good provided n_s is sufficiently large). As a kind of mean, the expectation values represent a diagnostic for the model as whole by mitigating any eccentricities arising from individual samples, and thus represent the proper series for comparison between model and system.

With these points in mind, it is perhaps somewhat puzzling that the authors rely almost exclusively on sample paths as their standard for comparing the agent-based model to historical data. Indeed, while there is an attempt to calculate a mean path (figure 4, Garnet *et al.* (2015)), it would appear from the 'noisiness' of the mean signal that the number of sample simulations used (n_s=10) is insufficient to provide a good approximation to the model's expectation values. Such an observation is valuable because it encourages us to think more deeply about the model the authors describe, and prompts additional questions concerning what the expectation values might be; whether they have any impact on the author's conclusions; and, indeed, if they can be used to draw a link back to the initial exponential growth/decay model of equation (1.2).

In what follows I consider these questions by examining the agent based model (ABM) of Garnet *et al.* from both an analytical and computational perspective. I begin by reviewing the model assumptions, and demonstrate that they motivate a two-population representation of the system (Section 2). Somewhat unexpectedly, I shall demonstrate that an exact closed form expression may derived for the model's expectation values, and use this expression to extend the authors' model to a regime for which interactions occur within a single population (ABM-SP). An exact method for calculating expectation values permits investigation into the effect of changing the merger completion probability d, and I shall show that the authors' recommendation of selecting d=0.4 may be interpreted as rejecting the hypothesis that an extended merger duration is needed for the model to fit historical data $B_H(t)$ (Section 3). Finally, I draw my results together by showing how the expectation values can be used to link both ABM and ABM-SP to the exponential model of equation (1.1) (Section 4).

2. Agent Based Model and Expectation Values

Since the agent-based model described by *Garnett et al.* operates according to probabilities, it is expedient to cast the processes involved in terms of expectation values, and for this reason I shall begin by introducing some notation. Adopting the authors' basic set-up, a given simulation has a total of n_T=2400 time-steps, each of duration Δt = 1 month, i.e., the time period modelled is $n_T\Delta t$ =2400 months, or 200 years. If we conduct a total of n_s simulations, then we can denote $B_i^n = B(t_n)$ as the number of banks at time step n for the ith simulation, where $i \in \{1,n_s\}$ and $t_n = n\Delta t$. Notice that based on the authors' historical data we have an initial condition $B_i^0 = B_H(0)$ = 1100 banks. In this way, the expectation value for the number of banks at time-step n, denoted $\langle B^n\rangle = \langle B(t_n)\rangle$, may be defined as

$$\langle B^n\rangle = \lim_{n_s\to\infty} \bar{B}(t_n;n_s), \quad \text{where} \quad \bar{B}(t_n;n_s) = \frac{1}{n_s}\sum_{i=1}^{i=n_s} B_i^n, \qquad (2.1)$$

is the mean value determined from n_s sample simulations. We now observe that the total bank population B_i^n comprises two sub-populations: a population of M_i^n 'merging' banks (banks engaged in a merger process); and a population of A_i^n 'active' banks (banks free to begin new merger processes). These sub-populations are related to the total bank population by the expressions

$$B_i^n = A_i^n + M_i^n, \quad \text{and} \quad \langle B^n \rangle = \langle A^n \rangle + \langle M^n \rangle, \tag{2.2}$$

where the expectation values $\langle A^n \rangle$ and $\langle M^n \rangle$ are defined in accordance with equation (2.1). Casting the system in this way allows us to describe the processes involved in ABM from the assumptions given by Garnett *et al.* (2015) as follows.

The authors state that the "number of opportunities to attempt to create a new bank with probability P [is] equal to the size of the existing population", where P in this instance is the creation probability per unit time a multiplied by Δt. Thus, at time-step n, the expected number of bank creations is $a\Delta t\langle B(t_n)\rangle$. Notice that since these newly created banks cannot be part of an existing merger process, they enter into the expected active population $\langle A^n \rangle$. Similarly, the expected number of active banks which fail at time-step n is $(b\Delta t)\langle A^n \rangle$. According to the model, any active banks which do not fail will begin a new merger process with probability $c\Delta t$ (Garnett *et al.* (2015)), i.e., the expected number of new merger processes beginning at t_n is $c\Delta t(1\text{-}b\Delta t)\langle A^n \rangle$. These are pairwise mergers (*two* 'active' banks merge to form *one* 'merging' bank), so will act to increase the merging bank population by the expected value of $c\Delta t(1\text{-}b\Delta t)\langle A^n \rangle/2$.

The population of merging banks in ABM may be treated in a similar fashion, and again the expected number of failures at t_n is $(b\Delta t)\langle M^n \rangle$. The authors also assume that any merging banks which survive (do not fail) can complete their merger process at end of the time-step with probability $d\Delta t$; hence, the expected number of merger processes finishing at t_n is $d\Delta t(1\text{-}b\Delta t)\langle M^n \rangle$. Following completion, these banks return to the active population $\langle A^n \rangle$ (Garnett *et al.* (2015)).

From the above description we see that the expected value of the total bank population within ABM evolves according to the coupled recurrence relations

$$\langle A^{n+1}\rangle = \langle A^n\rangle + a\Delta t\langle B^n\rangle - b\Delta t\langle A^n\rangle - c\Delta t(1-b\Delta t)\langle A^n\rangle + d\Delta t(1-b\Delta t)\langle M^n\rangle, \tag{2.3a}$$

$$\langle M^{n+1}\rangle = \langle M^n\rangle - b\Delta t\langle M^n\rangle + \tfrac{1}{2}c\Delta t(1-b\Delta t)\langle A^n\rangle - d\Delta t(1-b\Delta t)\langle M^n\rangle, \tag{2.3b}$$

and hence by the vector equation

$$\langle \boldsymbol{B}^{n+1}\rangle = (I + \Lambda\Delta t)\cdot\langle \boldsymbol{B}^n\rangle, \tag{2.4}$$

where the total bank population vector $\langle \mathbf{B}^n\rangle$, identity matrix I, and rate matrix Λ are defined

$$\langle \boldsymbol{B}^n\rangle = \begin{bmatrix}\langle A^n\rangle\\ \langle M^n\rangle\end{bmatrix},\quad I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \text{and}\quad \Lambda = \begin{bmatrix}(a-b-c(1-b\Delta t)) & (a+d(1-b\Delta t))\\ \tfrac{1}{2}c(1-b\Delta t) & -(b+d(1-b\Delta t))\end{bmatrix}. \tag{2.5}$$

Remarkably, since the elements of Λ are all constants, we can therefore write down an exact expression for the expectation values of ABM at time t^n, that is,

$$\langle \boldsymbol{B}^n\rangle = (I + \Lambda\Delta t)^n\cdot\langle \boldsymbol{B}^0\rangle, \tag{2.6}$$

where $\langle \mathbf{B}^0\rangle = (\langle A^0\rangle,\langle M^0\rangle)$ are the initial conditions $(\langle A^0\rangle,\langle M^0\rangle) = (1100,0)$. Such a result is powerful because it enables us to determine the expected number of banks from the authors' ABM system without having to perform a single simulation.

At this point I should emphasise that while I believe my interpretation of ABM to be accurate, it is possible that I have missed certain nuances of the scheme; for example, it may be that merging banks are not subject to failure. Should such differences exist, then their effect will be to slightly modify the exact form of the rate matrix Λ, so in principle it would still be possible to derive an expression for the expectation values as I have done above. In any case, we can test the model as it is described here by comparison with the historical data $B_H(t)$. Calculating a mean path from n_s = 1000 samples[1] suggests both that my expression for the expectation values is correct, and that my interpretation of ABM is, if not identical to that described by Garnett *et al.*, then extremely close, and well within the uncertainties present in the data (see figures 2 & 3).

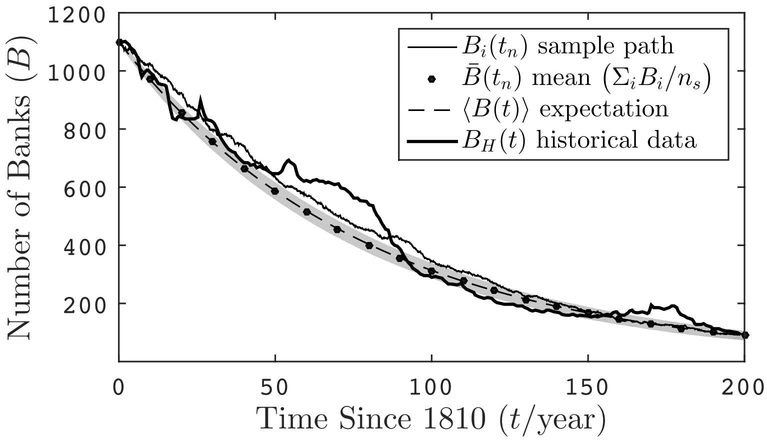

Figure 2. Comparison between the historical data (thick curve), a sample simulation from ABM (thin curve), the expectation values predicted by equation (2.6) (dashed curve), and the mean from n_s = 1000 sample simulations (circles). The grey shaded area represents the standard deviation of the sample simulations above and below the mean. Notice that there is excellent agreement between my analytic solution for the expectation value and the mean. Similarly, the expectation value represents a 'good fit' to the historical data (cf. figure 3).

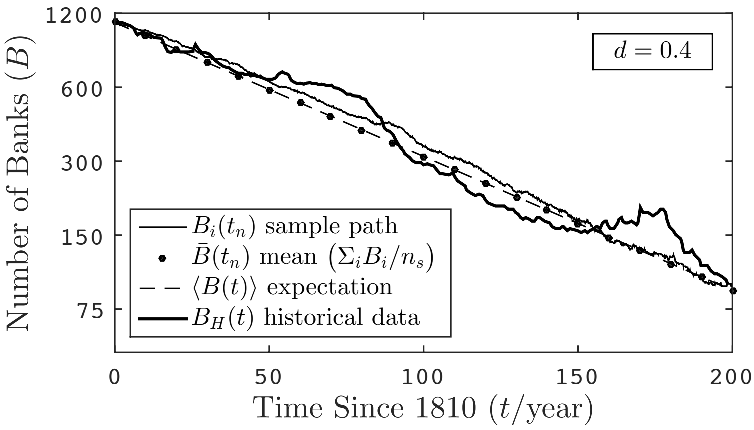

Figure 3. Data from figure 2 plotted with a logarithmic ordinate axis to indicate overall trends.

[1] For the purpose of tracking merger history, Garnett *et al.* employ a numerical scheme which interacts with a graph database. However, here I am free to exploit the indistinguishable nature of the bank agents, and implement a C++ algorithm which yields a favourable increase of speed relative to the authors' solver [n_s = 1000 simulations complete in a little under 30s on a MacBook Pro with 8GB of RAM].

Since the vector form of equation (2.6) is somewhat opaque, we can make progress in uncovering the effects of model parameters by summing together equations (2.3a) and (2.3b), thus,

$$\langle B^{n+1}\rangle = \langle B^n\rangle + a\Delta t\langle B^n\rangle - b\Delta t\langle B^n\rangle - \frac{1}{2}c\Delta t(1-b\Delta t)\left(1-\frac{\langle M^n\rangle}{\langle B^n\rangle}\right)\langle B^n\rangle. \quad (2.7)$$

Two observations now follow. First, we find that bank creation occurs according to a 'birth' term $a\Delta t\langle B(t_n)\rangle$ (see, e.g. Murray (2002)), raising the interesting historical question of why such creation should function as a birth process; perhaps one way to interpret bank creation, therefore, is as the probability that any given bank succeeds in founding a new bank. [Note: here the authors appear mistaken when they state that creations in the model are "not a 'birth' process" (Garnett *et al.* (2015).] Second, both the failure term and merger terms act to reduce the total bank population much like the 'death' rate in the exponential growth/decay model of equation (1.1). The real novelty of ABM, therefore, derives from merger duration effects in the parameter d, which maintain a merging population $\langle M^n\rangle$, and act to suppress the rate at which new merger processes begin by the factor $(1-\langle M^n\rangle/\langle B^n\rangle)$. When d is close to unity we expect $\langle M^n\rangle/\langle B^n\rangle<<1$, in which case the effects of merger duration are essentially negligible; conversely, when d is very small we expect larger values for $\langle M^n\rangle/\langle B^n\rangle$ (possibly of order unity), and a consequent reduction in bank merger rates.

Notice that taking d=1 represents the situation whereby all merger processes finish at the *end* of a simulation time-step (see equation (2.9) below); the closely related situation whereby all merger processes finish *within* a simulation time-step corresponds to equation (2.7) in the limit $\langle M^n\rangle\to 0$, that is,

$$\langle B^{n+1}\rangle = (1+\Delta t\lambda)\langle B^n\rangle, \text{ where } \lambda = a - b - \frac{1}{2}c(1-b\Delta t). \quad (2.8)$$

This equation has solution (cf. equation (2.6))

$$\langle B^n\rangle = (1+\Delta t\lambda)^n\langle B^0\rangle, \text{ with } \langle B^0\rangle = 1100, \quad (2.9)$$

so that—rather ironically perhaps—when $\langle M^n\rangle/\langle B^n\rangle << 1$ the decline in the population of British banks as predicted by the expectation values of ABM is described in terms of a law of (negative) compound interest ($\lambda<0$).

Observe that equations (2.8) and (2.9) are the single population analogues of equations (2.4) and (2.6); we therefore adopt equation (2.8) as the basis for postulating a complementary single population agent-based model ABM-SP. The mechanisms operating in ABM-SP are taken to be identical to those operating in ABM, and are implemented in the same way, but with the assumption that all merger processes conclude within a single time-step. For this reason ABM-SP is simply the authors' scheme ABM operating at the end of the merger duration spectrum. Indeed, it may be shown that in the absence of other effects the expected number of time-steps taken for a merger to complete in the ABM simulation is

$$\langle n\rangle = \frac{1}{d} \geq 1, \tag{2.10}$$

whereas ABM-SP is free from d dependence and assumes $\langle n\rangle < 1$. In more concrete terms, therefore, there exists a closer mechanistic correspondence between ABM-SP and ABM(d=1), than (say) there is between ABM(d=1) and ABM(d=0.001).

By comparing the simulation outputs from both ABM and ABM-SP (figures 2 & 3 and figures 4 & 5 respectively), one can see that in either case almost identical expectation values are obtained, alongside comparable statistical properties as represented by the mean data and standard deviations from n_s = 1000 samples. [Note: the sample paths plotted in these figures *are* rather different, highlighting one of the pitfalls of using individual samples as the standard for model comparison (see Introduction).] Given such similarities, therefore, it is appropriate to examine the impact of the merger duration probability d in more detail.

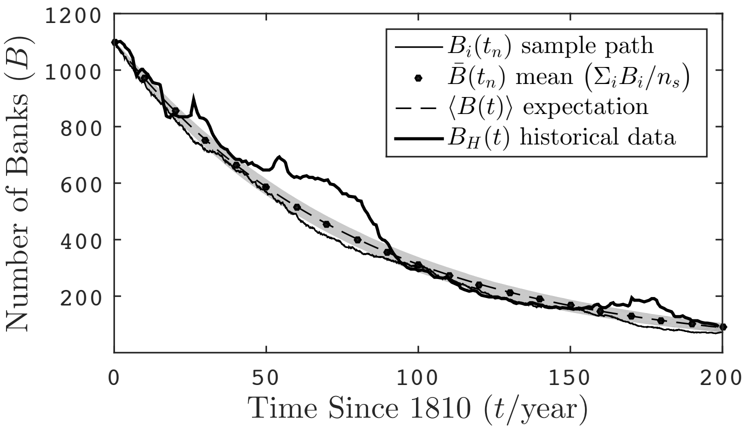

Figure 4. Comparison between the historical data (thick curve), a sample simulation from ABM-SP (thin curve), the expectation values predicted by equation (2.9) (dashed curve), and the mean from n_s = 1000 sample simulations (circles). The grey shaded area represents the standard deviation of the sample simulations above and below the mean. This figure should be compared with the ABM data plotted in figure 2.

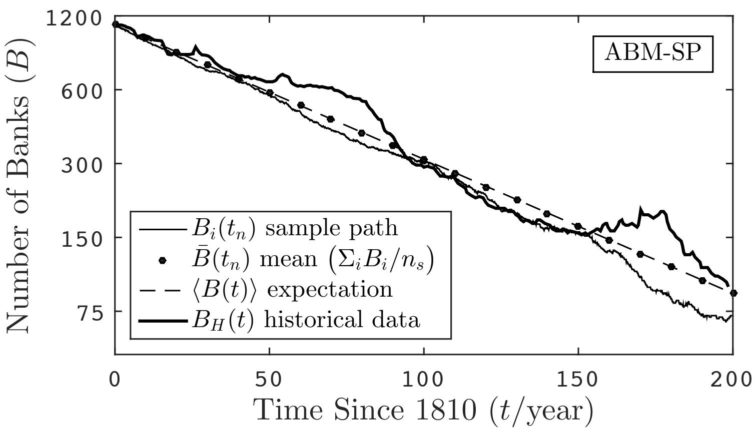

Figure 5. Data from figure 3a plotted with a logarithmic ordinate axis (cf. figure 3).

3. Impact of Merger Completion Probability

The authors consider the effect of changing the merger completion probability *d* towards the end of their paper using sample simulations (figure 7, Garnett *et al.* (2015)), and here we can contribute to their analysis by means of the newly derived expectation values discussed in the previous section. A brief investigation confirms their observation that for *d*<0.1 the model no longer fits the observed data. From equations (2.7) and (2.10) we can see why this is the case more formally: for small values of *d* (<0.1) the expected number of time-steps for a merger to complete will exceed approximately <*n*>=9, or 9 months (~1 year); this has the effect of maintaining a relatively large merging population $\langle M^n \rangle$, and hence a suppressed bank merger rate as described by the $(1-\langle M^n \rangle / \langle B^n \rangle)$ term in equation (2.7). Notice that this provides a concise explanation for the reduced maximum size of the bank 'merger trees' at the end of a given simulation (Garnett *et al.*(2015)).

When d>0.1 all the sample simulation paths cluster around the historical data; as the authors state, ABM "fits the observed data when the probability is between 0.1 and 1, with a value of 0.4 providing a good fit" (Garnett *et al.*). By extension, therefore, and with reference to figures 3, here we can supplement the set of acceptable d values with the authors' model running in the ABM-SP regime. Since the authors rely on sample paths, however, it is doubtful whether we are really justified in taking d=0.4 as the best fit. After all, the better fit of d=0.4 may simply be a consequence of the particular sample simulations in question (see Introduction). We are thus faced with the following problem: which value for the probability *should* be used in ABM?

One method of answering this question is to look at the fractional percentage difference between the model's expectation values for different values of $d\in[0.1,1]$. Here our analytical expressions are of great use, particularly because ABM-SP is not dependent on d, and may therefore be employed as a reference series. Indeed, we may write the fractional percentage difference $f(t_n,d)$ at time t_n between the expectation values from the two versions of the model as

$$f(t_n;d) = \left(\frac{\langle B(t_n;d)\rangle - \langle B(t_n;*)\rangle}{\langle B(t_n;*)\rangle}\right)\times 100, \tag{3.1}$$

where $\langle B(t_n;d\rangle$ corresponds to ABM with $d\in[0.1,1]$, and $\langle B(t_n;*\rangle$ are the values for ABM-SP, which has no d dependence. From figure 6 we see that this difference reaches a maximum of less than 3.5% over the simulation duration (2400 time-steps); in the particular case of d=0.4, the two versions of the model agree to within 0.8%.

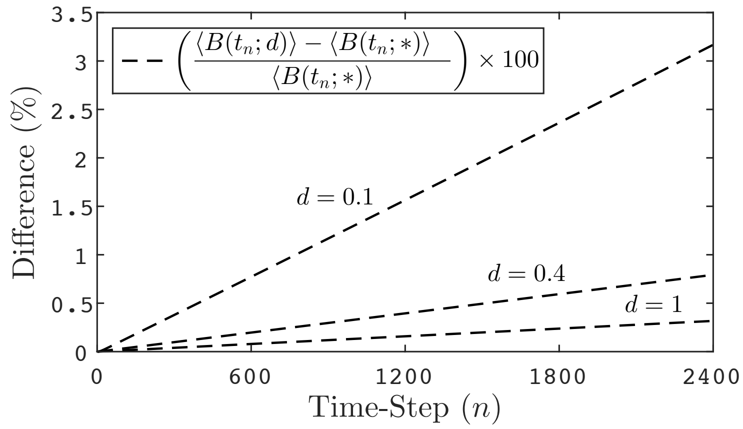

Figure 6. Fractional percentage difference between expectation values for ABM and ABM-SP over the simulation duration (2400 time-steps). The expectation values for ABM are dependent on the relevant *d*, and denoted here by ‹$B(t_n;d)$›; expectation values for ABM-SP have no *d* dependence, and are denoted ‹$B(t_n;*)$›.

When dealing with a complex system subject to large uncertainties within both the real data series and model parameters, the inclusion of a term (d=0.4) which affects expectation values by less than 0.8% requires very careful justification. By recommending a value of d=0.4∈[0.1,1], therefore, what the authors' are suggesting is that d be chosen such that it has negligible impact on model outcomes (‹M^n›/‹B^n›<<1, see section 2), i.e., that the hypothesis of merger duration impacting on organisational demography should be rejected.

Rejecting the hypothesis that the merger completion probability impacts on model outcomes is equivalent preferring the ABM-SP regime to the ABM model in its more general form. Asserting such a preference is good news for the authors' agent-based modelling approach for at least two reasons. First, since we are investigating a complex system for which paucity of modelling assumptions is at a premium, a move which reduces the number of system parameters at no cost to the model's predictive capability is extremely desirable. This is particularly true for ABM due to the way in which system parameters are determined. Indeed, one of the unique properties of the authors' carefully compiled data set is that they succeed in breaking down overall bank population statistics into a three distinct series (figure 4, Garnett *et al.* (2015)): a creation-series detailing instances of bank creation (cf. a); a failure-series detailing instances of bank failure (cf. b), and a merger-series detailing instances of bank merger (cf. c). Thus, while for the ABM model a set of *four* unknown probabilities $\{a,b,c,d\}$ must be calculated from *three* sets of data, in the ABM-SP model the situation more satisfactory: each of the three series may be used to determine their corresponding probability parameter a, b, and c.

The second reason for preferring the ABM-SP version of the authors' model concerns the verisimilitude of the merger duration process more generally. In particular, rather than stating that all merger processes conclude *within* a time-step, we can instead conceive of the ABM-SP system as meaning that banks are free to begin merger processes at any time (regardless of whether they are already engaged in merger), with the probability c reflecting the rate at which mergers do in fact occur. It seems plausible that such an interpretation is a more realistic representation of real life bank amalgamation processes than the restrictive system described by ABM (see Section 2).

4. Linking the Exponential Decay Model to the Agent-Based Model

Our discussion in the preceding section has demonstrated that by recommending d=0.4, the authors agent-based model ABM is in its essential features equivalent to the same model running in the ABM-SP regime, and predicts expectation values for the bank population $\langle B(t_n)\rangle$ that differ to those of ABM-SP by less than 0.8%. By employing the newly derived expressions for bank population expectation values (equation (2.8)), we are thus in an excellent position to link the authors' agent-based model to the exponential growth/decay model of equation (1.1). Indeed, using the notation

$$\Delta\langle B^n\rangle = \langle B^{n+1}\rangle - \langle B^n\rangle, \qquad (4.1)$$

for the the change in the expected bank population between time-steps n and n+1, the law of negative compound interest given by equation (2.8) may be written as

$$\frac{\Delta\langle B^n\rangle}{\Delta t} = \lambda\langle B^n\rangle, \quad \text{where} \quad \lambda = a - b - \frac{1}{2}c(1 - b\Delta t), \tag{4.2}$$

so that in the limit $\Delta t \rightarrow 0$ we recover the differential equation

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = \lambda B(t), \quad \text{where} \quad \lambda = a - b - \frac{1}{2}c, \tag{4.3}$$

i.e., the exponential model of equation (1.1) with 'birth' and 'death' rates $X = a$ and $Y = (b + c/2)$ respectively. Underneath the stochasticity of the agent-based implementation, therefore, both ABM(d=0.4) and ABM-SP are direct discrete analogues of the exponential growth/decay system we first introduced.

Though equation (1.1) and equation (4.3) are in some sense identical, it should be emphasised that the authors' agent-based modelling approach to studying the population dynamics of the British banking system has two major benefits over simply 'writing down' a differential equation. First, by following the complex systems methodology of carefully stating modelling assumptions, we have been able to test the hypothesis that a merger duration term is needed for the model to adequately describe the historical data. That this hypothesis has been discarded is beside the point: we would not be in a position to know that the term is unnecessary without conducting the study; indeed, its rejection from the model tells us something valuable about which interactions within the system are significant. Second, by analysing the historical data into creation, failure, and merger statistics, agent-based modelling supports the authors' mechanistic description of system processes, and emphasises the comparable importance of bank amalgamations (i.e., not simply bank failures) in determining the evolution of the total bank population.

5. Summary and Conclusions

The recent study by Garnett *et al.* (2015) describes two models for the decline in the British bank population $B(t)$ since 1810: a simple exponential model based on 'birth' and 'death' rates; and an agent-based model (ABM) in which organisational demography evolves according to probabilistic rules governing interactions between indistinguishable bank agents. Here we have shown that the probabilistic nature of the agent-based model may be exploited to derive exact analytical expressions for the model's expectation values $\langle B(t_n)\rangle$. Such expressions are useful for at least two reasons: first, they allow us to explore the meaning of agent interactions; and second, expectation values are the proper series for comparison between model output and historical data.

In contrast to the authors' description, our analytical results reveal that bank creation *is* a 'birth' process, raising the important historical question of why creation occurs in proportion to the number of existing banks. In addition, our expressions show that the affect of merger duration, controlled by the completion probability d, is to suppress overall creation rates, suggesting an extension of the ABM model to a regime whereby mergers are permitted to conclude within an individual simulation time-step (i.e., no d term), which we call ABM-SP. Using the authors' parameter values, we have seen that ABM and ABM-SP both match the historical data closely, prompting further investigation into the impact of the duration term in d. By conducting such an investigation, we have found that the authors' recommended value d=0.4 leads to a difference between the expectation values of ABM(d=0.4) and ABM-SP of less than 0.8%, i.e., that the hypothesis of merger duration impacting on the evolution of the bank population be should be rejected. Given that paucity of assumptions is at a premium when studying complex systems, simplifying the model by discarding a term which has no discernable effect on the model's predictive capability is good news for the authors' agent-based modelling process. Further, a plausible interpretation of ABM-SP may be to say that merger processes do not necessarily finish within a simulation time-step, rather that banks are not required to conclude a given merger before initiating a new amalgamation process, with the rate c reflecting the number of mergers which actually occur. Such an interpretation may reflect real life bank amalgamation processes more accurately than the proscriptive scheme of ABM.

The analytic expressions for the expectation values have the additional virtue that they allow us to draw a link between the exponential model and the agent-based system. In particular, underneath the stochastic interactions ('noise') present within ABM, we find that the agent-based model is a discrete implementation of the exponential decay law, with the latter's 'death' rate subdivided into failure and merger terms. This supports the authors' conclusion that bank amalgamation (in addition to failure) is a key driver in bank organisational demography (Garnett *et al.*, (2015)); however, it also leads us somewhat ironically to the observation that the decline in the British bank population since 1810 obeys a law of negative compound interest.

References:

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